The Orbits of a Charged Particle round an Electric and Magnetic Nucleus.

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The following communication is formally a complement to one published in the 'Proceedings' of the Society* on "The Effect of the Magneton on the Scattering of α-Rays."

In the present paper the more general case of a central positively charged nucleus possessing mass and a magnetic moment is considered. The case is treated as if the mass of the nucleus is so large compared with that of the revolving particle that it may be regarded as fixed. It is, therefore, not directly applicable when the revolving body is an α-particle except in cases where the central mass is large compared with that of the hydrogen atom. It is shown later what modification is needed when the motion of the nucleus is not large enough to affect its magnetic quality. The former paper was suggested by certain theories relating to the scattering of α and β-particles by matter. In the present, however, the chief interest lies in the discussion of the nature and properties of the various orbits, more especially of such as do not extend to infinity, or as they may be called "local orbits." In both cases the motion in the equatorial plane of the magneton alone is considered.

Orbits of Positive Particle.

The following specification of the system will be used:—

Charge on the nucleus $= ne$ magnetic $= nec$ static units.

Charge on the moving particle $= n'e$ magnetic $= n'ec$ static units, e being the positive electronic charge and $n' = 2$ for an α-particle.

Moment of magneton $=$ M.

Mass of moving particle $= m$.

Suppose the magneton has its N pole above the paper. Then its field will always turn the particle to the left of its path, and, ρ denoting the radius of curvature,

$$\frac{mv^2}{\rho} = (\mathrm{M}n'ev - nn'e^2c^2p)\frac{1}{r^3}.$$

The equation of energy gives

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mathrm{V}^2 - \frac{nn'e^2c^2}{r},$$

* Series A, vol. 90, p. 356.

in which V denotes the velocity at infinity, supposed parallel to the initial line of θ.

These equations may be written

$$\frac{1}{\rho} = \left\{\frac{a^2}{\sqrt{(1-bu)}} - \frac{\frac{1}{2}pb}{1-bu}\right\}u^3,$$

$$mv^2 = m\mathrm{V}^2(1-bu),$$

where $$a^2 = \frac{\mathrm{M}n'e}{m\mathrm{V}}, \qquad b = \frac{2nn'e^2c^2}{m\mathrm{V}^2}, \qquad u = \frac{1}{r},$$

both a and b being of the dimensions of a line.

Since $$\frac{1}{\rho} = \frac{1}{r}\cdot\frac{dp}{dr} = u^3\frac{dp}{du},$$

$$\frac{dp}{du} = \frac{1}{2}\frac{b}{1-bu}p - \frac{a^2}{\sqrt{(1-bu)}},$$

whence $$p = \frac{p'-a^2u}{\sqrt{(1-bu)}}, \tag{1}$$

where p' is the value of p at infinity.

Apsidal Distances.—These are given by $p = r = 1/u$, or

$$1-bu-u^2(p'-a^2u)^2 = 0. \tag{2}$$

It will simplify matters if we use a unit of length $= a$; in other words, write u for au, p for p/a, and b for b/a.

The apsidal distances are then given by the roots of

$$1-bu-u^2(p'-u)^2 = 0.$$

Tangents to Orbit from O.—When the tangents exist $p = 0$,

$$u = p' \text{ (or, in ordinary units, } au = p'/a). \tag{3}$$

Points of Inflection.—When these exist $dp/dr = 0$, or ∞.

This condition gives $$u = \frac{2-p'b}{b},$$

or, in ordinary units, $$r = \frac{a^2b}{2a^2-p'b}. \tag{4}$$

They are, therefore, non-existent if $p' > 2a^2/b$ (in ordinary units), also $r > b$, therefore $p' > a^2/b$, or points of inflection exist if $p' > a^2/b < 2a^2/b$.

Since $p = r^2(d\theta/ds)$, it follows that

$$d\theta = \frac{p'-u}{\sqrt{\{1-bu-u^2(p'-u)^2\}}}du$$

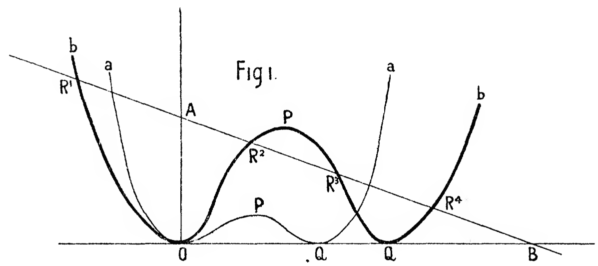
$$= \frac{p'-u}{\sqrt{-\{(u-\alpha)(u-\beta)(u-\gamma)(u-\delta)\}}}du, \tag{5}$$

where α, β, γ, δ are the roots of the apsidal equation supposed in ascending order of magnitude. It will be shown later that α is always negative.

The apsidal distances are given by the biquadratic

$$1-bu-u^2(p'-u)^2 = 0.$$

The roots of this are determined by the intersections of the fixed straight line $y = 1 - bu$ and the curve $y = u^2(p-u)^2$, where now p is written for p'. These vary for given a, b, as p' is changed from $-\infty$ to $+\infty$. The curve is symmetrical with respect to a point at $u = \frac{1}{2}p$, where there is a hump, and has two branches stretching to infinity as in fig. 1, a, b.



If p' is negative, the curve is reversed with respect to OA, whilst the line AB remains unchanged.

As p' increases from 0 to ∞ the curve, while keeping its general shape, grows, the hump at P rising and Q moving out to the right. During this growth the hump may touch the line AB on the inside.

If so, it will again touch for a larger p' on the outside of the line, and between these the hump will cut the line in two points as in fig. 1, b, and the equation has four real roots.

If, on the contrary, the hump does not touch AB as it grows there will never be more than one real positive root. It is clear then that there is always one real negative root and at least one positive, and in certain conditions three real positive ones.

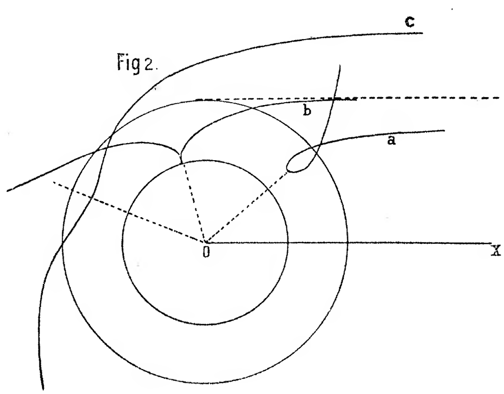
The integration for θ extends from $u = 0$ to the smallest positive root, and in case of three real positive roots, there is another orbit corresponding to integration between the two largest. In other words we get orbits stretching to infinity for all values of p', and, under certain circumstances, orbits always at a finite distance. It is then necessary to determine the criterion for the existence of these "local" orbits.

If p' is negative the curve is reversed relatively to OA and there is always one and only one positive real root. In other words, only orbits extending to infinity can exist. As will be seen immediately, the top of the hump grows

very much more rapidly with increasing u than the ordinate of the line. Consequently, while for $p'=0$ there will be one negative root only, there is always a value of p' beyond which the hump will cut the line and the equation will have three real negative roots. The negative roots do not of course refer to any actual orbits, but the nature of the reality of the roots will affect the transformation by which the integral giving θ is evaluated.

In fig. 1, Q corresponds to the tangent from the centre to the orbit, the points R to the apses, and B is a point of no velocity.

In the case where (p' being positive) the hump never touches AB, as u increases from 0 it comes to Q before the apse, so long as OQ is less than OB, *i.e.* so long as $p < 1/b$. In this case there are actual tangents from the centre to the orbit, of lengths given by (3). All particles therefore projected from ∞ in lines distant from the nucleus less than a^2/b proceed in orbits with tangents from the centre, and may therefore form loops. When Q is at B, *i.e.* $p = 1/b$, the contact points of the tangents and the apse coincide and the velocity is zero. In other words, when the line of projection is at a distance a^2/b cm. from the centre the orbit has a cusp at a distance from the centre $= b$ cm. As Q advances beyond B, the root lies to the left of B, *i.e.* u comes to an apse before a possible tangent, and the orbits have no tangents from the centre. On the other hand points of inflection now exist so long as $p'<2/b$ and when $p' = 2/b$ the point of inflection is at infinity. For $p'>2/b$ they do not exist. In fig. 2, *a*, *b*, *c* illustrate the nature of these orbits when nearest the nucleus. The figure is only diagrammatic. The inner circle has radius b, the outer radius a^2/b.



In the case here considered there can be no combined atomic system.

In the second case the hump may cut the line AB in two points. Suppose it first touches it for a value of $p' = p_1$, and touches it outside for the value $p' = p_2$. For $p' = p_1$ the orbits are similar to those in the case just considered

with tangents from O, that is, in general, loops. But as p' becomes nearly equal to p_1 we have the roots β and γ of equation (5) becoming equal, and θ becoming very large as u approaches the neighbourhood of β. In other words as the particle approaches from infinity and gets within a distance a/β it makes a large number of almost circular revolutions and then again unravels itself until it reaches an apse by means of a loop. When $p' = p_1$ the single orbit becomes two—one from infinity ultimately revolving in a circle of radius a/β and unravelling itself after an infinite time by a corresponding path to infinity; the other with one loop, having tangents from the centre approaching asymptotically from the inside to the same circle. It clearly corresponds to a circular orbit with instability. When p' lies between p_1 and p_2 there are two branches—one due to integration from $u = 0$ to $u = \beta$ and the second from $u = \gamma$ to $u = \delta$. The first is from infinity, has no tangent from O, and a single apse a/β. The second is a local orbit with apsidal distances a/γ and a/δ. As p' increases from p_1 to p_2, Q moves out to B and then beyond. In other words, for $p' < a^2/b$ Q lies between the apses, and there are tangents from O, *i.e.* scrolls with loops. When Q is at B the loops degenerate to cusps. For larger values of p' up to p_2, Q is outside B, there are no tangents, but, as before, points of inflection occur. As p' approaches p_2 the roots γ, δ, tend to equality, θ becomes very large. That is the orbit again becomes a circle when $p' = p_2$ with a radius a/δ. This radius is greater than the cuspidal case and less than the unstable circular orbit above. It is therefore stable, *i.e.* a slight disturbance does not send the particle to infinity.

It is clearly important to know the conditions which discriminate between these two cases. To do this it is necessary as a preliminary to discuss some points as to the nature of the hump.

The equation to the curve is

$$y = u^2 (p-u)^2.$$

The peak of the hump is given by

$$u = \tfrac{1}{2}p, \qquad y = p^4/16,$$

and its locus therefore is $y = u^4$, *i.e.* the same as the curve for $p = 0$.

$$dy/du = 2u(p-u)(p-2u).$$

There are points of inflection given by

$$p^2 - 6pu + 6u^2 = 0,$$

whence

$$u = \tfrac{1}{2}p(1 \pm 1/\sqrt{3}). \tag{6}$$

For the point farthest out

$$u = \tfrac{1}{2}p(1 + 1/\sqrt{3}).$$

At the point of inflection

$$\frac{dy}{du} = -\frac{p^3}{3\sqrt{3}},$$

$$y = \frac{p^4}{36} = \frac{4}{9} \times \text{height of the peak.}$$

If p' be increased from a given value

$$dy/dp = 2u^2(p-u) = + \text{ so long as } u<p.$$

That is, if p be increased, the hump of the new curve lies wholly outside the old one.

Discrimination of Cases i *and* ii.—It is clear that if the inclination of the tangent at the point of inflection is $< \tan^{-1}b$, the curve can never cut the line AB in more than two points. Consider the case where the point of inflection lies on the line and touches it, that is, choose p' so that the point of inflection is on it, and then b, so that the corresponding $dy/du = -b$.

p' is given by

$$p^4/36 = 1-\tfrac{1}{2}bp(1+1/\sqrt{3}),$$

also

$$-b = -p^3/3\sqrt{3}.$$

Eliminating p, the condition for the possibility is

$$b = \{4(2-\sqrt{3})/\sqrt{3}\}^{\frac{3}{4}},$$

or, using ordinary units,

$$b = \{4(2-\sqrt{3})/\sqrt{3}\}^{\frac{3}{4}}a = 0{\cdot}6976921\ldots a,$$

and then

$$p'/a = \{4(2-\sqrt{3})/\sqrt{3}\}^{\frac{1}{4}}\sqrt{3} = 1{\cdot}536202\ldots$$

Suppose p', b/a, have the above values. Then, with p' greater, dy/du at the point of inflection increases. With p' less, the hump cannot cut the line in two points, and with p' greater, the humps always lie outside the above critical one, and therefore will not cut in more than one point. If, however, the tangent at the point of inflection when it is on the line is $> \tan^{-1}b$, it will cut in two points. Hence the discrimination is that for two points on the hump (or four roots in all), dy/du of the point of inflection when on the line must be numerically $>b$. In other words—

$$\begin{array}{ll} \text{for Case i} & b> \\ \quad\text{Case ii} & b< \end{array} \Big| \; 0{\cdot}6976921\ldots a. \tag{7}$$

In case ii, with $b/a < 0{\cdot}697\ldots$, the apsidal equation will have four real roots for values of p' between two limiting values p_1, p_2, corresponding to the two cases when the hump touches the line (a) below AB and (b) above.

In other words, p_1, p_2, must be such as to make the apsidal equation have equal roots. Consequently

$$\begin{aligned} 1-bu-u^2(p-u)^2 &= 0 \\ -b-2u(p-u)(p-2u) &= 0. \end{aligned}$$

The equation for the p values, found by the elimination of u, is found to be

$$4bp^5-4p^4+b^3p^3+30b^2p^2-9bp+64+27b^4/4 = 0, \tag{8}$$

and the course of the elimination gives for the corresponding double roots of the apsidal equation

$$u = \frac{24b-3b^2p-4p^3}{18b^2-8p^2+4bp^3} \tag{9}$$

in which p is a root of the above quintic.

Since the sum of the four roots of the apsidal equation is $2p$ and their product -1, the sum of the other two is $2(p-u)$ and their product $-1/u^2$, where u is the double root (9). Hence the other two roots are

$$p-u \pm \sqrt{\{(p-u)^2+1/a^2\}}. \tag{10}$$

The diagrams show that there will always be one negative root of (8) (as is clear from the equation itself) and two real positive roots if $b/a < 0{\cdot}697\ldots$, and no more.

The Angular Co-ordinate in the Orbit.—Equation (5) gives

$$\theta = \int_0^u \frac{(p'-u)\,du}{\sqrt{-\{(u-\alpha)(u-\beta)(u-\gamma)(u-\delta)\}}} = \int_0^u \frac{p'-u}{\sqrt{-\mathrm{Y}}}\,du \quad \text{(say)}$$

for the infinite branch, with apsidal angle given by the upper limit $u = \beta$.

When a local orbit exists, *i.e.* γ, δ, real

$$\theta = \int_\gamma^u \frac{p'-u}{\sqrt{-\mathrm{Y}}}\,du,$$

with apsidal angle given by the upper limit $u = \delta$ and the tangential angle by $u = p$ (when $p < \delta$).

In a particular case, the first step is to solve the apsidal equation, and then apply a transformation by the known methods to reduce the elliptic integral to a canonical form. As the chief interest of the present communication is in connection with local orbits, *i.e.* orbits lying wholly within a finite distance of the centre, it will be necessary only to carry out this work for the case where all the roots are real, and p' positive. For the infinite branches $u > 0 < \beta$, using the quadric transformation, write*

$$u = \frac{\alpha(\delta-\beta)+\delta(\beta-\alpha)\sin^2\phi}{\delta-\beta+(\beta-\alpha)\sin^2\phi},$$

* Notation and formulæ of Briot and Bouquet, 'Théorie des Fonctions Elliptiques,' p. 428.

$$k^2 = \frac{(\beta-\alpha)(\delta-\gamma)}{(\gamma-\alpha)(\delta-\beta)} \quad \text{(anharmonic ratio of } R_1, R_2, R_3, R_4 \text{ in fig. 1)},$$

whence

$$\sin^2\phi = \frac{(\delta-\beta)(u-\alpha)}{(\beta-\alpha)(\delta-u)} \quad \text{(inverse ratio of } R_1, R_2, P, R_4).$$

When $u = 0$, $\qquad \sin\phi = \sqrt{-\frac{\alpha}{\delta}\left(\frac{\delta-\beta}{\beta-\alpha}\right)} = \sin\phi_0$ (say)

" $\quad u = \beta, \qquad \phi = \pi/2.$

Also from the apsidal equation

$$p' = \tfrac{1}{2}(\alpha+\beta+\gamma+\delta).$$

Making the substitutions

$$\theta = \frac{\alpha+\beta+\gamma-\delta}{\sqrt{(\gamma-\alpha)(\delta-\beta)}}\int_{\phi_0}^{\phi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}$$
$$+2\frac{(\delta-\alpha)(\delta-\beta)}{(\beta-\alpha)\sqrt{(\gamma-\alpha)(\delta-\beta)}}\int_{\phi_0}^{\phi}\frac{d\phi}{(n+\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}} \qquad (11)$$

where $\qquad n = \frac{\delta-\beta}{\beta-\alpha}, \qquad \sin\phi_0 = \sqrt{-\frac{n\alpha}{\delta}}.$

The apsidal angle Θ is given by $\phi = \pi/2$, or

$$\Theta = \frac{\alpha+\beta+\gamma-\delta}{\sqrt{(\gamma-\alpha)(\delta-\beta)}}\left\{F(\pi/2)-F(\phi_0)\right\}$$
$$+2\frac{(\delta-\alpha)(\delta-\beta)}{(\beta-\alpha)\sqrt{(\gamma-\alpha)(\delta-\beta)}}\{\Pi(\pi/2)-\Pi(\phi_0)\}.$$

In the cuspidal case, where $p = b$, $\alpha+\beta+\gamma-\delta = 0$, and the first elliptic integral F does not enter.

For the local orbits the integration is from $u = \gamma$ up to $u = \delta$, k has the same value as above, but the transformations are given by

$$u = \frac{\gamma(\delta-\beta)-\beta(\delta-\gamma)\sin^2\phi}{\delta-\beta-(\delta-\gamma)\sin^2\phi}, \qquad \sin^2\phi = \frac{(\delta-\beta)(u-\gamma)}{(\delta-\gamma)(u-\beta)},$$

with $\phi = 0$ when $u = \gamma$, and $\phi = \pi/2$ when $u = \delta$,

$$\theta = \frac{\alpha-\beta+\gamma+\delta}{\sqrt{(\gamma-\alpha)(\delta-\beta)}}F(\phi)+2\frac{(\gamma-\beta)(\delta-\beta)}{(\delta-\gamma)\sqrt{(\gamma-\alpha)(\delta-\beta)}}\Pi(\phi), \qquad (12)$$

where now $\qquad n = -\frac{\delta-\beta}{\delta-\gamma},$

and the apsidal angle is given by $\phi = \pi/2$.

For the cuspidal case, the largest root is given by $\delta = 1/b$, and the others, therefore, by the cubic

$$bu^3-u^2+b^2 = 0.$$

To solve this, put $u = \frac{1}{2} b\sqrt{3} \sec\phi$, then

$$\cos 3\phi = -\frac{3\sqrt{3}}{2} b^2 = -\cos 3\chi$$

where $3\chi < \pi/2$.

Then
$$\phi = 2n\frac{\pi}{3} \pm \left(\frac{\pi}{3} - \chi\right)$$

and the roots are

$$\alpha = \frac{b\sqrt{3}}{2} \sec(\pi - \chi) = -\frac{b\sqrt{3}}{2} \sec\chi,$$

$$\beta = \frac{b\sqrt{3}}{2} \sec(\pi/3 - \chi),$$

$$\gamma = \frac{b\sqrt{3}}{2} \sec(\pi/3 + \chi),$$

with
$$\delta = \frac{1}{b} = \frac{3b\sqrt{3}}{2} \sec 3\chi.$$

For real roots, therefore, $b < \sqrt{\frac{2}{3\sqrt{3}}} < 0{\cdot}62$. This is less than the general limit for four real roots, as it clearly should be. If b is larger, χ is a pure imaginary $= \chi' i$, where $\cosh 3\chi' = 3\sqrt{3}b^2/2$, and we have the roots

$$\alpha = -\frac{b\sqrt{3}}{2} \Big/ \cosh\chi' \qquad \text{(real)}$$

$$\left.\begin{matrix}\beta \\ \gamma\end{matrix}\right| = \frac{b\sqrt{3}}{4} \frac{\sqrt{3}\cosh\chi' \pm i \sinh\chi'}{\cos^2(\pi/3) - \cosh^2\chi'}.$$

The above give for all roots real

$$k^2 = \frac{\cos(\pi/6 - \chi)\cos^3(\pi/3 + \chi)}{\cos(\pi/6 + \chi)\cos^3(\pi/3 - \chi)},$$

$$n = \frac{4}{\sqrt{3}} \frac{\cos^3(\pi/3 - \chi)\cos\chi}{\cos(\pi/6 + \chi)\cos 3\chi} \text{ for the infinite branch,}$$

$$n = -\frac{\cos^2(\pi/3 - \chi)}{\cos^2(\pi/3 + \chi)} \text{ for local orbit,}$$

whilst the equation for the local orbit is

$$\theta = \sqrt{\frac{2\cot(\pi/3 - \chi)}{\cos\pi/6}} \mathrm{F}(\phi) + \frac{\sin 2\chi}{\cos^2(\pi/3 + \chi)} \sqrt{\{2\cos(\pi/6)\cot(\pi/3 - \chi)\}}\, \Pi(\phi).$$

The limiting cases, however, *i.e.* corresponding to equal roots of the apsidal equation, do not require elliptic functions. The first corresponds to $\beta = \gamma$ and belongs to an orbit from infinity, the other to $\gamma = \delta$ and belongs to a local circular orbit. In the first

$$\theta = \int_0^u \frac{p - u}{(\beta - u)\sqrt{(u - \alpha)(\delta - u)}} du,$$

whence, remembering that $p = \frac{1}{2}(\alpha+2\beta+\delta)$,

$$\theta = \sin^{-1}\frac{\alpha+\delta}{\delta-\alpha} - \sin^{-1}\frac{\delta+\alpha-2u}{\delta-\alpha} + \frac{1}{2}\frac{\delta+\alpha}{\sqrt{(\beta-\alpha)(\delta-\beta)}}$$

$$\times \log_e \frac{\beta}{\beta-u}\,\frac{\beta(\delta+\alpha)/2-\alpha\delta-[\beta-(\delta+\alpha)/2]u+\sqrt{(\beta-\alpha)(\delta-\beta)(u-\alpha)(\delta-u)}}{\beta(\delta+\alpha)/2-\alpha\delta+\sqrt{-\alpha\delta(\beta-\alpha)(\delta-\beta)}}, \tag{13}$$

and for the local orbit, integrating from u to δ,

$$\theta = -\frac{1}{2}\pi + \sin^{-1}\frac{2u-\alpha-\delta}{\delta-\alpha} + \frac{1}{2}\frac{\delta+\alpha}{\sqrt{(\beta-\alpha)(\delta-\beta)}}$$

$$\times \log_e \frac{\delta-\beta}{u-\beta}\,\frac{\beta(\delta+\alpha)/2-\alpha\delta-[\beta-(\delta+\alpha)/2]u+\sqrt{(\beta-\alpha)(\delta-\beta)(u-\alpha)(\delta-u)}}{\frac{1}{2}(\delta+\alpha)(\delta+\beta)-\delta(\beta+\alpha)}. \tag{14}$$

In the latter the angular co-ordinate is measured from the apse $u = \delta$. In both, of course, the apsidal angle is infinite.

In the particular case of three equal roots, where $b = \left\{4\frac{2-\sqrt{3}}{\sqrt{3}}\right\}^{\frac{3}{4}}$, the orbit from infinity becomes

$$\theta = \frac{\delta+\alpha}{\delta-\alpha}\left\{\sqrt{\frac{u-\alpha}{\delta-u}} - \sqrt{-\frac{\alpha}{\delta}}\right\} + \sin^{-1}\frac{\delta+\alpha}{\delta-\alpha} - \sin^{-1}\frac{\delta+\alpha-2u}{\delta-\alpha},$$

whilst the local orbit merges in the asymptotic circle $u = \delta$, or $r = 0{\cdot}8254a$.

In the above δ is the abscissa of the point of inflection on the hump, *i.e.* $\delta = \frac{1}{2}p(1+1/\sqrt{3}) = \frac{1}{2}(\sqrt{3}+1)b^{\frac{1}{3}}$. The sum of all the roots of the apsidal equation is $2p$. Hence

$$\alpha = 2p-3\delta = -\frac{\sqrt{3}}{2}(\sqrt{3}-1)b^{\frac{1}{3}}.$$

Substituting these values, the equation to this orbit becomes

$$\theta = \frac{\sqrt{3}-1}{2}\left\{\sqrt{\frac{u+\frac{1}{2}\sqrt{3}(\sqrt{3}-1)b^{\frac{1}{3}}}{\frac{1}{2}(\sqrt{3}+1)b^{\frac{1}{3}}-u}} - (\sqrt{3}-1)\sqrt{\frac{\sqrt{3}}{2}}\right\}$$

$$+\sin^{-1}\frac{2u-(\sqrt{3}-1)b^{\frac{1}{3}}}{2b^{\frac{1}{3}}} + 21^\circ\,28',$$

or putting in numerical values, and returning to ordinary units

$$\theta = \left\{0{\cdot}1165\sqrt{\frac{au+0{\cdot}5623}{1{\cdot}2115-au}} + 0{\cdot}0400\right\}\pi + \sin^{-1}(1{\cdot}1275\,au - 0{\cdot}3660).$$

It is interesting to note that this orbit is in plan altogether independent of the constitution of the nucleus—*i.e.* of M, n, n, e. Its size alone depends on it. With a given constitution, the definite value of b/a required determines the energy of the system—which then gives the value of a (see below).

The foregoing may be illustrated by a numerical example. In order

that local orbits may be possible it is necessary, as we have seen, that $b/a < 0{\cdot}6976\ldots$. Let us take then $b/a = 0{\cdot}5$.

The apsidal equation is then

$$1-\tfrac{1}{2}u-u^2(p-u)^2 = 0.$$

The limiting values of p for these orbits are the two positive roots of

$$2p^5-4p^4+\tfrac{1}{8}p^3+\tfrac{15}{2}p^2-48p+64\tfrac{27}{64} = 0.$$

They are $p = 1{\cdot}7067,\ 2{\cdot}0324$.

The apsidal equation gives the following roots:—

For $p = 1{\cdot}7067$.	For $p = 2{\cdot}0324$.
$\alpha = -0{\cdot}5059$	$\alpha = -0{\cdot}4462$
$\beta = 1{\cdot}0330$	$\beta = 0{\cdot}5802$
$\gamma = 1{\cdot}0330$	$\gamma = 1{\cdot}9654$
$\delta = 1{\cdot}8530$	$\delta = 1{\cdot}9654$

The orbits for $p = 1{\cdot}7067$ are—for the path from infinity

$$\theta = 0{\cdot}5995\log_e \frac{1{\cdot}0330}{1{\cdot}0330-u}\cdot\frac{1{\cdot}4538-0{\cdot}3199u+\sqrt{(u+0{\cdot}5059)(1{\cdot}8530-u)}}{2{\cdot}4220}$$

$$+\sin^{-1}\frac{u-0{\cdot}6735}{1{\cdot}1794}+0{\cdot}5709,$$

and for the local orbit, the same expression with

$$\frac{0{\cdot}8197}{u-1{\cdot}0330} \quad \text{in place of} \quad \frac{1{\cdot}0330}{1{\cdot}0330-u},$$

in which in ordinary units $u = a/r$; for $p = 2{\cdot}0324$, the local orbit is a circle $r = a/\delta = 0{\cdot}5084a$.

For the cuspidal orbit $p = 2$ and the roots of the apsidal equation are $\alpha = -0{\cdot}4516$, $\beta = 0{\cdot}5970$, $\gamma = 1{\cdot}8546$, $\delta = 2$.

The equations for the orbits are then

(*a*) Orbit to infinity—

$$\theta = 3{\cdot}6471\,\{\Pi(n,k,\phi)-\Pi(n,k,\phi_0)\},$$

with $\quad k^2 = 0{\cdot}04712, \quad n = 1{\cdot}3380, \quad \phi_0 = 33^\circ\ 20'$,

(*b*) Local orbit—

$$\theta = 1{\cdot}5600\,\mathrm{F}(\phi)+13{\cdot}492\,\Pi(n,k,\phi),$$

with $\quad k^2 = 0{\cdot}04712, \quad n = -9{\cdot}6494$.

The apsidal angles are respectively

$$\Theta = 102^\circ\ 43', \qquad 7^\circ\ 25'.$$

The angle between two successive cusps in the local orbits is therefore $14^\circ\ 50'$.

With the same value of b we may take $p = 1{\cdot}85$—about midway between the two limits—for an example of looped orbits. For this value of b, however, the cusped orbit is so close to the inner limit that the inflexional orbits cover a very small region. With $p = 1{\cdot}85$, the apsidal roots are $\alpha = 0{\cdot}4782$, $\beta = 0{\cdot}7016$, $\gamma = 1{\cdot}5358$, $\delta = 1{\cdot}9408$. The equation to the local orbit is

$$\theta = 1{\cdot}454\,\mathrm{F}(\phi) + 3{\cdot}231\,\Pi(n, k, \phi),$$

with $k^2 = 0{\cdot}1914, \quad n = -3{\cdot}060, \quad \sin^2\phi = 3{\cdot}060\,\dfrac{u - 1{\cdot}5358}{u - 0{\cdot}7016}.$

The apsidal angle is $15°\ 11'$ and the tangent from the centre makes an angle $2°\ 53'$ with the line to the inner apse. The apsidal distances are $0{\cdot}651a$ and $0{\cdot}513a$ and the length of the tangent, $0{\cdot}540a$.

Orbits of Negative Particle.

In this case the force due to the magnetic field is to the right of the path, and the electric force is attractive. Hence, with the same notation as before,

$$mv^2/\rho = (-\mathrm{M}ev + nn'e^2c^2p)\,u^3,$$
$$\tfrac{1}{2}mv^2 = \mathrm{C} + nn'e^2c^2/r,$$

in which, however, C will be negative for cases where the energy is less than that due to a fall from infinity. Write $\mathrm{C} = \frac{1}{2}m\mathrm{V}^2 f$ where $f = +1$ for paths from infinity and $f = -1$ when the energy is less than that from infinity.

Then as before it may be shown that

$$p = \frac{p' + a^2u}{\sqrt{(bu + f)}}.$$

The apses are given by the biquadratic

$$f + bu - u^2(p' + a^2u)^2 = 0\,;$$

or, taking the unit of length $= a$, by

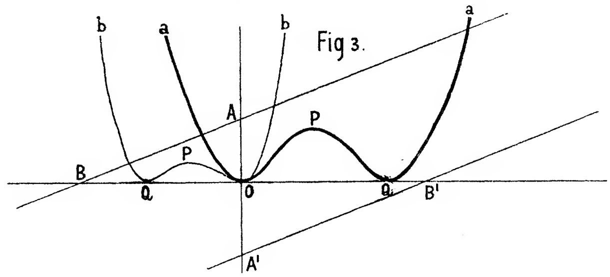
$$f + bu - u^2(p + u)^2 = 0.$$

Case i.—*Energy greater than that from Infinity, or* $f = +1$.—The roots are given by the intersections of $y = 1 + bu$ and $y = u^2(p + u)^2$.

Fig. 3b corresponds to p' positive. There is always one, and only one positive root—the largest root δ. In other words there is always an infinite branch with one apse, and no tangent from the centre, for Q lies to the left of O.

Fig. 3a corresponds to p' negative. There is always one real positive root, and may be two more if the hump can cut the line. Since the locus of the top of the hump is $y = u^4$ it must cut $y = 1 + bu$ somewhere, since its ordinate increases with u much faster than that of the line.

There will thus be local orbits for values of p' greater than some critical value, viz., for the case of two equal roots. This corresponds to the case



of the positive particle, in which the infinite orbit and the local orbit have a common asymptotic circle. The conditions for this case can be written down at once from the equations for the case of the positive particle already obtained by writing $-b$ for b and $-p'$ for p'.

Thus the points of inflection are from equation (4) given by

$$u = p' - 2/b,$$

and exist so long as $p' > 2/b$, or $p' > 2a^2/b$ in ordinary units.

There are therefore none with p' negative.

The conditions for four real roots of the apsidal equation, *i.e.* for a local orbit, are found from equation (8) to be the negative root of the unaltered equation, viz.,

$$4bp^5 - 4p^4 + b^3p^3 + 30b^2p^2 - 96bp + 64 + 27b^4/4 = 0.$$

The angular co-ordinate θ is given by

$$\theta = \int \frac{(u+p')\,du}{\sqrt{-\{(u-\alpha)(u-\beta)(u-\gamma)(u-\delta)\}}},$$

in which it is to be remembered that p' is essentially negative. For the infinite branch the integration extends from $u = 0$ up to $u = \beta$, and for the local orbit from $u = \gamma$ to $u = \delta$. The orbit $\beta = \gamma$ is expressible in logarithmic functions, with asymptotic circles, but there is none corresponding to $\gamma = \delta$. Further, in this motion, contrary to that of the positive particle, u may have infinitely great values—that is, local orbits exist in which the revolving particle may be as close as desired to the centre even with a velocity greater than that due to a fall from infinity—supposed, of course, that it does not come close enough to interfere with the structure of the magneton or the nucleus. The existence of these combinations with extreme energy has its clear suggestions in connection with the theory of the inner electrons of atomic theory. They cannot exist with electric attractions only. Naturally, also, if the velocities are comparable with that of light the whole theory is inapplicable.

The signs of the p' in local orbits refer to directions of motion round the centre.

In this case i, the transformations to reduce the elliptic integrals to the canonical form are the same as in the former case.

Case ii.—*Energy less than that from Infinity, or* $f = -1$.—The apsidal equation is

$$-1+bu-u^2(p'+a^2u)^2 = 0.$$

In this case, in the diagram of fig. 3 the line is displaced to A′B′, and the curve (b) corresponds to p' positive. There is clearly no real root, or two only. The limit for the two is found by the condition that the curve touches the line, which gives a critical value for p'. Since dy/dp for the curve is positive for u positive the curve grows steeper with increasing p. Hence for two real roots p' must be less than this critical value. If the apsidal equation be written

$$1-bu+u^2(p'+a^2u)^2 = 0,$$

it can be transformed from the former equation by writing $-p'i$ for p' and a^2i for a^2. If the equation for the critical p be thus transformed from equation (8) the corresponding equation for this case becomes

$$4bp^5+4p^4-b^3p^3+30b^2p^2+96bp+64-27b^4/4 = 0. \tag{15}$$

with the corresponding

$$u = \frac{24b+3b^2p-4p^3}{18b^2+8p^2+4bp^3}.$$

In order, then, to have orbits with p' positive, *i.e.* rotating anti-clockwise, it is necessary for the above equation to have a positive root. It is clear from fig. 3b that there cannot be more than one. Hence $p = 0$ and $p = \infty$ must make the expression change sign, and this it cannot do unless $64-27b^4/4$ is negative. Hence the required condition is

$$27b^4/4>64,$$

or

$$b>4/3^{\frac{3}{4}}>1{\cdot}754766\ldots,$$

with $b = 4/3^{\frac{3}{4}}$, $p = 0$, and the two equal roots of the apsidal equation are

$$u = 4/3b = 3^{-\frac{1}{4}}$$

or

$$r = 3^{\frac{1}{4}}a \text{ in ordinary units.}$$

$$= 1{\cdot}316074\ldots a.$$

This gives a circular orbit. When b is greater than the above the equation (15) has one real positive root, say p_1. Then for any value of p' less than this, there results a local orbit with two apsidal distances, and no

tangent from the centre. The orbit, in fact, encloses the nucleus without any loops.

Fig. $3a$ corresponds to p' negative, or a clockwise direction of motion. Here clearly there may be no real positive roots, or two or four, to the apsidal equation. As $-p'$ increases, the right-hand branch of the curve must cut the line for some value of $-p'$ greater than the largest root of equation (15), which equation always has at least one real negative root. It does not necessarily follow that it has three real ones—or that the apsidal equation has four real positive roots—unless b satisfies some condition. This condition is determined in a similar way to the corresponding case for the positive particle. The line must be steeper than a certain amount, *i.e.* $b >$ a critical value. This is determined by the condition that the point of inflection must lie on the line and the line touches the curve there. But here it is necessary to take the point of inflection which is nearer to the axis of y. The condition is then found to be that

$$b > \left(4\frac{2+\sqrt{3}}{\sqrt{3}}\right)^{\frac{3}{4}} > 5{\cdot}030195\ldots,$$

which may be compared with the corresponding value in the former case

$$b < \left(4\frac{2-\sqrt{3}}{\sqrt{3}}\right)^{\frac{3}{4}} < 0{\cdot}697\ldots.$$

The cases of all roots imaginary occur when the line does not cut the curve of $p' = 0$. The condition of this has been found for p' positive above. If $b > 4 \times 3^{-\frac{3}{4}}$ there will always be at least two roots.

The figure shows that with some particular value of p, viz. $-1/b$, Q will fall on B. For p' less than this value there are no tangents to the orbit. For $p' = -1/b$ there will be cusps for the smaller of the u-roots, *i.e.* the cusps point outwards. For $-p' > 1/b$, Q lies between the two largest roots, tangents from O exist, and the orbit has loops. When there are four roots, the orbit with the largest roots has tangents from O, the other not: that is the inner orbit has loops, the outer not.

Points of inflection are given by

$$u = 2/b - p' = 2\mathrm{OB} - \mathrm{OQ}.$$

When therefore Q is on the right of B, $u < \mathrm{OB}$, and as u is always greater than OB points of inflection do not exist. That is orbits with cusps or loops have no points of inflection. When Q is on the left of B, $u > \mathrm{OB}$, and it is not impossible for such to exist. To determine whether they actually exist it is necessary to find whether the value of u lies between the roots γ, δ, or α, β: *i.e.* whether the value of $-1 + bu - u^2(p-u)^2$ is positive when $2/b - p$ is inserted for u, and $p' < 1/b = 1/b - x$ (say). It becomes

$bx-(1/b+x)^2x^2 = x\{b-x(1/b+x)\}$. When u is small this is $+$. That is the orbits near the cuspidal ones have points of inflection, and they will exist so long as $2/b-p'>\gamma$.

Effect of Inertia of the Nucleus.—In the foregoing the nucleus has been treated as fixed. This will introduce no appreciable error in the case where the revolving particle is an electron. The error, however, may be considerable for α revolving particles, even when the mass of the nucleus itself is large, and the theory is inapplicable as it stands to systems comparable with H or He atoms. Throughout the assumption is made that the velocities are small compared with that of light, and that energy losses due to accelerations are negligible. Under these circumstances the electric forces on the nucleus and its satellite are equal and opposite, and so far as they are concerned the common centre of gravity may be regarded as fixed. The case of the magnetic forces, however, is not so clear. That there is a resultant reaction equal to the magnetic force on the moving particle we naturally assume with Newton's third law; but it does not follow that this reaction is concentrated on the magneton, and that there is nothing on the medium itself or its boundary. The following considerations, however, show that for the problem under consideration, the reaction on the nucleus is equal and opposite to that on the moving particle—viz.: $\mathrm{M}ev/r^3$ along the normal. The moving particle produces a magnetic field in circles round its direction of motion. Let H denote this force at O and let ON be the perpendicular from O on the line of motion. Then (decomposing M into its elements m, $-m$, at distance l) it is easy to show that the force on the magneton is along ON and is MH/ON. But if v^2/c^2 is negligible $\mathrm{H} = ev\mathrm{ON}/r^3$. Hence the force is $\mathrm{M}ev/r^3$, the same in magnitude as that on the moving particle. To this extent, therefore, we may be justified in regarding the centre of gravity as fixed. We make also another supposition that the motion of the magneton does not affect its properties.

If r denote the distance of the moving particle from the centre of gravity, the distance from the nucleus is $(1+m/m')r = kr$(say). The forces along the normal are, therefore, $\mathrm{M}n'ev/k^3r^3$ due to the magneton, and $nn'e^2c^2p/k^2r^3$ due to the component of the electric force. Also the energy equation gives

$$k\tfrac{1}{2}mv^2 = k\tfrac{1}{2}m\mathrm{V}^2-\frac{nn'e^2c^2}{kr},$$

or

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mathrm{V}^2-\frac{nn'e^2c^2}{k^2r}.$$

The theory is therefore precisely similar to that developed above, in which a^2 is replaced by a^2/k^3 and b by b/k^2.

Circumstances of Projection.—These give the velocity of projection, the distance from the nucleus, and the angle the direction of projection makes with this. The velocity gives, with distance, the energy $\frac{1}{2}mV^2$ of the system. From this a^2 and b can be calculated.

If u' be the inverse distance, p the perpendicular, and ϕ the direction of motion, $pu' = \cos\phi$. The corresponding value of p' is then found from equation (1), viz.:—

$$p' = a^2u' + \frac{\sqrt{(1-bu')}}{u'}\cos\phi.$$

The actual orbit is then determined by the methods already developed.

Numerical Values.—Let E denote the total energy of the system, *i.e.* the kinetic energy when the potential is zero, or in case of energy less than from infinity, the potential energy when the kinetic is zero. The quantities now determining the orbits are the two:—

$$a^2 = \frac{\mathrm{M}n'e}{\sqrt{(2m)}\sqrt{\mathrm{E}}}, \qquad b = \frac{nn'e^2c^2}{\mathrm{E}}.$$

We shall regard M as consisting of a whole number N of magnetons and take

$$\mathrm{M} = 1{\cdot}6 \times 10^{-21}\,\mathrm{N}.$$
$$e = 1{\cdot}55 \times 10^{-20}.$$
$$ec = 4{\cdot}65 \times 10^{-10}.$$
$$m \text{ for H} = 1{\cdot}6 \times 10^{-24}.$$
$$m \text{ for electron} = 0{\cdot}84 \times 10^{-27}.$$

Energy of α-particle moving with velocity of light $= 2{\cdot}88 \times 10^{-3}$.
Energy of mass of electron moving with velocity of light $= 3{\cdot}78 \times 10^{-7}$.

Further it will be convenient to measure E above, as a fraction x of these quantities, so that

For α-particle $\qquad \mathrm{E} = 2{\cdot}88 \times 10^{-3}x.$

For electron $\qquad \mathrm{E} = 3{\cdot}78 \times 10^{-7}x.$

With these values we get—

(i) For α-particles—

$$a^2 = 1{\cdot}29\,\mathrm{N}n'x^{-\frac{1}{2}}10^{-28}. \qquad a = 1{\cdot}13\,\mathrm{N}^{\frac{1}{2}}n'^{\frac{1}{2}}x^{-\frac{1}{4}}10^{-14}.$$
$$b = 7{\cdot}50\,nn'x^{-1}10^{-17}.$$
$$a^2/b = 1{\cdot}72\,\mathrm{N}n^{-1}x^{\frac{1}{2}}10^{-12}. \qquad b/a = 6{\cdot}63\,\mathrm{N}^{-\frac{1}{2}}nn'^{\frac{1}{2}}x^{-\frac{3}{4}}10^{-3}.$$

The volume a^4/b is independent of the energy. It corresponds to that of a sphere of radius $3{\cdot}75\,(\mathrm{N}^2n'/n)^{\frac{1}{3}}10^{-14}$.

(ii) For electron—

$$a^2 = 0{\cdot}98\,\mathrm{N}n'x^{-\frac{1}{2}}10^{-24}, \qquad a = 0{\cdot}99\,\mathrm{N}^{\frac{1}{2}}n'^{\frac{1}{2}}x^{-\frac{1}{4}}10^{-12},$$

$$b = 5{\cdot}72\,nn'x^{-1}10^{-13},$$

$$a^2/b = 1{\cdot}73\,\mathrm{N}n^{-1}x^{\frac{1}{2}}10^{-12}, \qquad b/a = 0{\cdot}577\,\mathrm{N}^{-\frac{1}{2}}nn'^{\frac{1}{2}}x^{-\frac{3}{4}},$$

whilst the radius of the sphere corresponding to a^4/b is $7{\cdot}43(\mathrm{N}^2n'/n)^{\frac{1}{3}}10^{-13}$.

In order that combined atoms consisting of one α-particle ($n'=2$) may exist, $b/a < 0{\cdot}697$—whence

$$x > 0{\cdot}0032\,(n^2/\mathrm{N})^{\frac{2}{3}},$$

corresponding to a velocity when free $> 0{\cdot}056\,(n^2/\mathrm{N})^{\frac{1}{3}}c$; or the sizes of possible orbits are of the order of $a^2/b > (\mathrm{N}^2/n)^{\frac{1}{3}}10^{-13}$.

The cusped orbit for the limiting case above—*i.e.* the nearest in—is $b=0{\cdot}47\,(\mathrm{N}^2/n)^{\frac{1}{3}}10^{-13}$. We should expect the velocity of the α-ray, when expelled, to be about $(n^2/\mathrm{N})^{\frac{1}{3}}/20$ of light. That of Ra is about one-twentieth. All that can be expected from the present theory is information as to the order of effects—dealing as it does with one revolving particle only. It is interesting, therefore, to find that the orbits which correspond to radioactive instability (asymptotic circles) give values of velocity of emission comparable with those observed.

In case of revolving electrons there are two types. First, that in which the energy is less than that from infinity, in which, therefore, a combined atom may be regarded as of a more permanent type, secondly, that in which the atom retains in combination the electron, even though the energy be greater than that from infinity—the β-radiation analogue. The condition for the latter type is $b/a > 5{\cdot}030$. This gives

$$x < 0{\cdot}0557\,(n^2/\mathrm{N})^{\frac{2}{3}},$$

$$\text{Free velocity} < 0{\cdot}236\,(n^2/\mathrm{N})^{\frac{1}{3}}c,$$

$$a^2/b < 4{\cdot}06\,(\mathrm{N}^2/n)^{\frac{1}{3}}10^{-13},$$

$$b > 1{\cdot}0\,(\mathrm{N}^2/n)^{\frac{1}{3}}10^{-11}.$$

[*Note, March* 20.—By a reference given by Dr. Allen my attention has been drawn to the fact that the problem of the motion of an electron round a magneton and electric charge has already been considered in its most general aspect by Störmer. I have not been able to see the original paper of the latter ('Christiania Forhandl.,' 1907), but he has given a *résumé* of his results in the 'Comptes Rendus' (see especially vol. 156 (1913), pp. 450, 536). The generality of his method, however, would not appear to be susceptible of easily giving a detailed discussion of the orbits themselves and does not cover the ground of the present paper.]